J

Mathematical Proofs

J.1 The Free Space Condition

$$\omega^a{}_b = -\kappa \epsilon^a{}_{bc} q^c$$

This fundamental condition is a solution of:

$$\boxed{R^a{}_b \wedge q^b = \omega^a{}_b \wedge T^b} \tag{J.1}$$

$$\left(D \wedge \omega^a{}_b\right) \wedge q^b = \omega^a{}_b \wedge \left(D \wedge q^b\right) \tag{J.2}$$

$$\left(d \wedge \omega^a{}_b\right) \wedge q^b + \left(\omega^a{}_c \wedge \omega^c{}_b\right) \wedge q^b = \omega^a{}_b \wedge \left(d \wedge q^b\right) + \omega^a{}_b \wedge \left(\omega^b{}_c \wedge q^c\right) \tag{J.3}$$

To Prove:

$$\left(d \wedge \omega^a{}_b\right) \wedge q^b = \omega^a{}_b \wedge \left(d \wedge q^b\right) \tag{J.4}$$

Proof for $a = 1$

$$\left(d \wedge \omega^1{}_2\right) \wedge q^2 + \left(d \wedge \omega^1{}_3\right) \wedge q^3 = \omega^1{}_2 \wedge \left(d \wedge q^2\right) + \omega^1{}_3 \wedge \left(d \wedge q^3\right) \tag{J.5}$$

Eqn. (J.5)

$$\omega^1{}_2 = -\kappa \epsilon^1{}_{23} q^3 = \kappa \epsilon_{123} q^3 = \kappa q^3 \tag{J.6}$$

$$\omega^1{}_3 = -\kappa \epsilon^1{}_{23} q^2 = \kappa \epsilon_{132} q^2 = -\kappa q^2 \tag{J.7}$$

i.e.

$$\left(d \wedge q^3\right) \wedge q^2 - \left(d \wedge q^2\right) \wedge q^3 = q^3 \wedge \left(d \wedge q^2\right) - q^2 \wedge \left(d \wedge q^3\right) \tag{J.8}$$

$$\begin{aligned} \implies \left(d \wedge q^3\right) \wedge q^2 &= -q^2 \wedge \left(d \wedge q^3\right) \\ -\left(d \wedge q^2\right) \wedge q^3 &= q^3 \wedge \left(d \wedge q^2\right) \end{aligned} \tag{J.9}$$

To prove

$$\left(\omega^a{}_c \wedge \omega^c{}_b\right) \wedge q^b = \omega^a{}_b \wedge \left(\omega^b{}_c \wedge q^c\right) \tag{J.10}$$

Proof for a=1, b=2, c=3;

$$\left(\omega^1{}_3 \wedge \omega^3{}_2\right) \wedge q^2 = \omega^1{}_2 \wedge \left(\omega^2{}_3 \wedge q^3\right) \tag{J.11}$$

where

$$\omega^1{}_2 = \kappa q^3; \;\; \omega^1{}_3 = -\kappa q^2$$
$$\omega^3{}_2 = -\kappa q^1; \;\; \omega^2{}_3 = \kappa q^1$$

therefore

$$\left(q^2 \wedge q^1\right) \wedge q^2 = -q^3 \wedge \left(q^1 \wedge q^3\right)$$

i.e.

$$\begin{aligned} q^3 \wedge q^2 &= -q^3 \wedge \left(-q^2\right) \\ &= q^3 \wedge q^2 \end{aligned} \tag{J.12}$$

For O(3) electrodynamics we choose:

$$\omega^a{}_b = -\frac{1}{2}\kappa \epsilon^a{}_{bc} q^c \tag{J.13}$$

in the structure relation:

$$D \wedge q^a = d \wedge q^a + \omega^a{}_b \wedge q^b \tag{J.14}$$

Proof For a=1:

$$D \wedge q^1 = d \wedge q^1 + \frac{1}{2}\left(\epsilon^1{}_{23} q^3 \wedge q^2 + \epsilon^1{}_{32} q^2 \wedge q^3\right) \tag{J.15}$$

$$\boxed{D \wedge q^1 = d \wedge q^1 + \kappa q^2 \wedge q^3}$$

In the O(3) circular complex basis this gives O(3) electrodynamics.

This allows the tetrad of the free field to be identified as the potential, and also the spin connection. O(3) electrodynamics is therefore a fundamental theory of general relativity.

J.2 The Tetrad Postulate

The tetrad postulate follows from the fact that a tensor is independent of the way it is written. The postulate follows from a consideration of the covariant derivative of a vector in two different bases. We denote these by J.16 and J.17. thus:

$$(DX)_1 = (DX)_2 \tag{J.16}$$

It follows that:

$$D_\nu q^a{}_\mu = 0. \tag{J.17}$$

For those interested a detailed proof is given as follows but eqn. (J.16) is enough to know where the tetrad postulate comes from.

Detailed Proof

In the coordinate basis (see Carroll(3.129))

$$\begin{aligned} DX &= (D_\mu X^\nu)\, dx^\mu \otimes \partial_\nu \\ &= \left(\partial_\mu X^\nu + \Gamma^\nu{}_{\mu\lambda}\, X^\lambda\right) dx^\mu \otimes \partial_\nu \end{aligned} \tag{J.18}$$

In the mixed basis:

$$\begin{aligned} DX &= (D_\mu X^a)\, dx^\mu \otimes \hat{e}_{(a)} \\ &= \left(\partial_\mu X^a + \omega^a{}_{\mu b} X^b\right) dx^\mu \otimes \hat{e}_{(a)} \end{aligned} \tag{J.19}$$

$$\begin{aligned} &= \left(\partial_\mu \left(q^a{}_\nu X^\nu\right) + \omega^a{}_{\mu b} q^b{}_\lambda X^\lambda\right) dx^\mu \otimes \left(q^\sigma{}_a \partial_\sigma\right) \\ &= q^\sigma{}_a \left(q^a{}_\nu \partial_\mu X^\nu + X^\nu \partial_\mu q^a{}_\nu + \omega^a{}_{\mu b} q^b{}_\lambda X^\lambda\right) dx^\mu \otimes \partial_\sigma \end{aligned} \tag{J.20}$$

where we have used the commutator rule. Now switch σ to μ and use:

$$q^\nu{}_a q^a{}_\nu = 1 \tag{J.21}$$

to obtain:

$$DX = \left(\partial_\mu X^\nu + q^\nu{}_a \partial_\mu q^a{}_\lambda X^\lambda + q^\nu{}_a q^b{}_\lambda \omega^a{}_{\mu b} X^\lambda\right) dx^\mu \otimes \partial_\nu \tag{J.22}$$

Now compare eqn. (J.18) and (J.22) to give:

$$\Gamma^\nu{}_{\mu\lambda} = q^\nu{}_a \partial_\mu q^a{}_\lambda + q^\nu{}_a q^b{}_\lambda \omega^a{}_{\mu b} \tag{J.23}$$

multiply both sides of eqn.(J.23) by $q^a{}_\nu$:

$$q^a{}_\nu \Gamma^\nu{}_{\mu\lambda} = \partial_\mu q^a{}_\lambda + q^b{}_\lambda \omega^a{}_{\mu b} \tag{J.24}$$

i.e.

$$\boxed{D_\mu q^a{}_\lambda = \partial_\mu q^a{}_\lambda + \omega^a{}_{\mu b} q^b{}_\lambda - \Gamma^\nu{}_{\mu\lambda}\, q^a{}_\nu = 0} \tag{J.25}$$

□

Eqn. (J.25) is known as **the tetrad postulate**, and is true for all connections.

Meaning of the Tetrad Postulate

The tetrad postulate means that the basis chosen for DX does not affect the result. The tetrad postulate originates in the definition of the tetrad itself:

$$V^a = q^a{}_\mu V^\mu \tag{J.26}$$

where a refers to the tangent spacetime and μ to the base manifold.

J.3 The Evans Lemma

The Evans Lemma is the direct result of **the tetrad postulate** of differential geometry:

$$\boxed{D_\mu q^a{}_\lambda = \partial_\mu q^a{}_\lambda + \omega^a{}_{\mu b} q^b{}_\lambda - \Gamma^\nu{}_{\mu\lambda}\, q^a{}_\nu = 0} \tag{J.27}$$

using the notation of the text. It follows from eqn. (J.27) that:

$$D^\mu (D_\mu q^a{}_\lambda) = \partial^\mu \left(D_\mu q^a{}_\lambda\right) = 0, \tag{J.28}$$

i.e.

$$\partial^\mu \left(\partial_\mu q^a{}_\lambda + \omega^a{}_{\mu b} q^b{}_\lambda - \Gamma^\nu{}_{\mu\lambda}\, q^a{}_\nu\right) = 0, \tag{J.29}$$

or

$$\Box q^a{}_\lambda = \partial^\mu \left(\Gamma^\nu{}_{\mu\lambda}\, q^a{}_\nu\right) - \partial^\mu \left(\omega^a{}_{\mu b} q^b{}_\lambda\right). \tag{J.30}$$

Define:

$$R q^a{}_\lambda := \partial^\mu \left(\Gamma^\nu{}_{\mu\lambda}\, q^a{}_\nu\right) - \partial^\mu \left(\omega^a{}_{\mu b} q^b{}_\lambda\right) \tag{J.31}$$

to obtain **the Evans Lemma**:

$$\boxed{\Box q^a{}_\lambda = R q^a{}_\lambda} \tag{J.32}$$